Identifiability of dynamic networks with noisy and noise-free nodes

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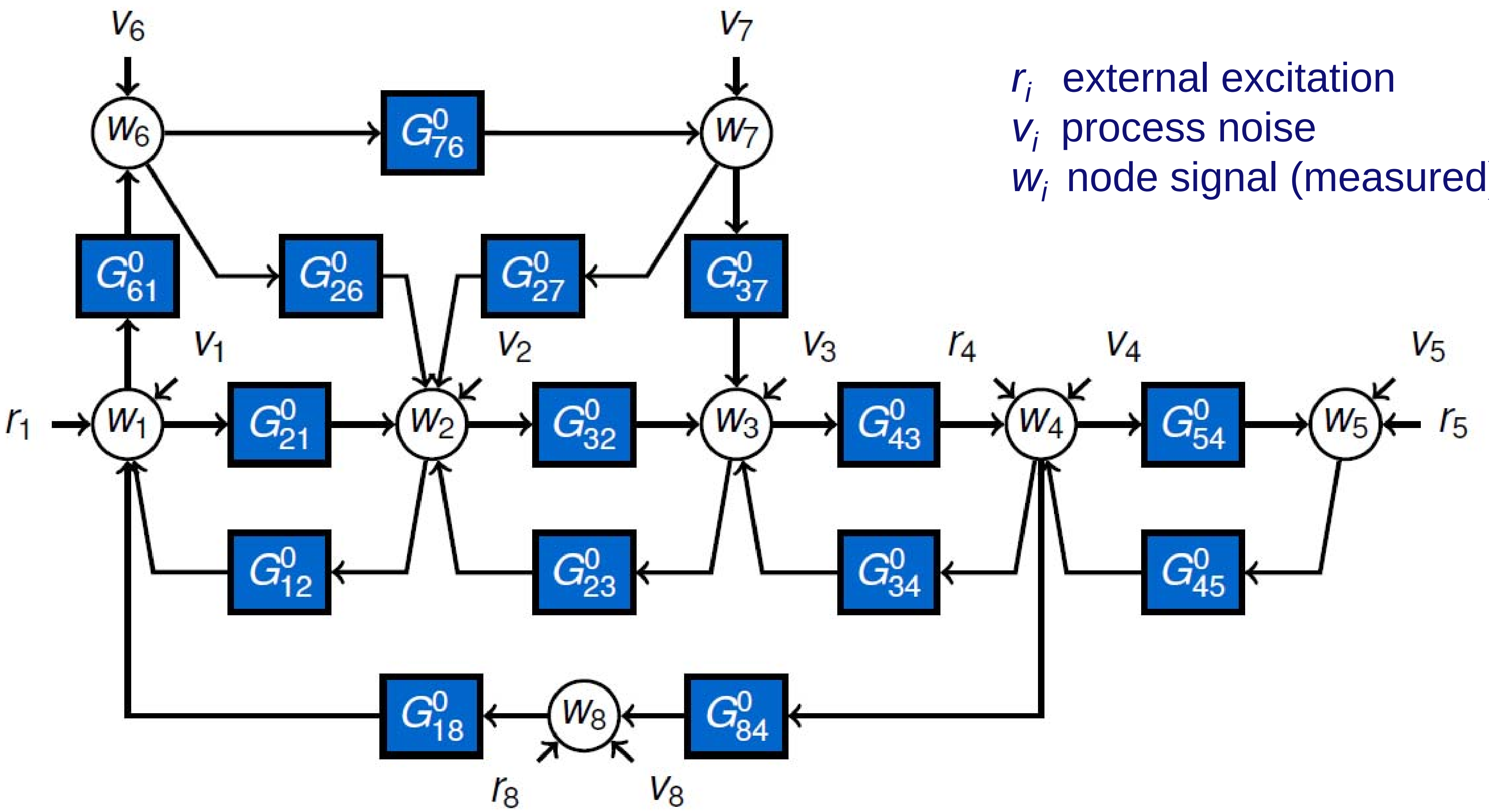
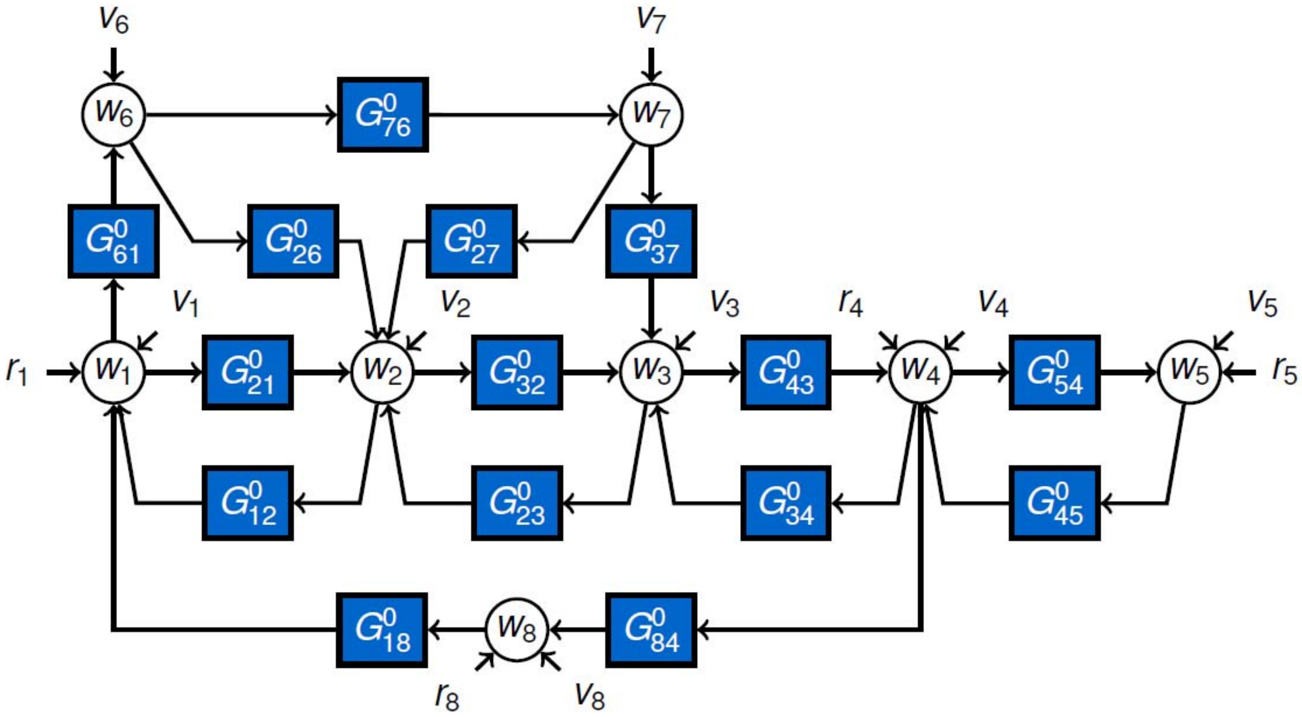
Where innovation starts

Dynamic network



r_i external excitation
v_i process noise
w_i node signal (measured)

Introduction – relevant identification questions

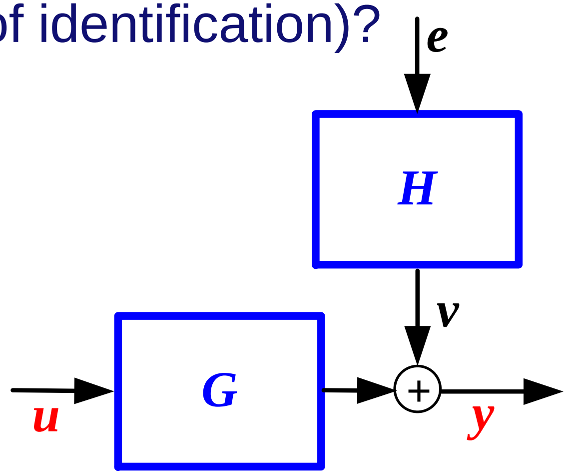


Question: Can the dynamics/topology of a network be *uniquely determined* from measured signals w_i, r_i ?

Question: Can different dynamic networks be *distinguished* from each other from measured signals w_i, r_i ?

Introduction

When are models essentially different (in view of identification)?



In classical PE identification:
Models are indistinguishable (from data) if their predictor filters are the same:

$$\hat{y}(t|t-1) = \underbrace{H(q)^{-1}G(q)}_{W_u(q)}\, u(t) + \underbrace{[1 - H(q)^{-1}]}_{W_y(q)}\, y(t)$$

(G_1, H_1) and (G_2, H_2) are indistinguishable iff

$$\begin{cases} H_1^{-1}G_1 = H_2^{-1}G_2 \\ 1 - H_1^{-1} = 1 - H_2^{-1} \end{cases} \Leftrightarrow \begin{cases} G_1 = G_2 \\ H_1 = H_2 \end{cases}$$

Introduction

For a parametrized model set (model structure):

$$\hat{y}(t|t-1;\theta) = \underbrace{H(q,\theta)^{-1}G(q,\theta)}_{W_u(q,\theta)}u(t) + \underbrace{[1-H(q,\theta)^{-1}]}_{W_y(q,\theta)}y(t) \qquad \theta \in \Theta$$

parameter values can be distinguished if

$$\left.\begin{array}{c} G(\theta_1) = G(\theta_2) \\ H(\theta_1) = H(\theta_2) \end{array}\right\} \Longrightarrow \theta_1 = \theta_2 \quad \text{for all } \theta \in \Theta$$

This property is generally known as the property of identifiability of the model structure

Introduction

So there are two different bijective mappings involved:

$$(W_y(q,\theta), W_u(q,\theta)) \Longleftrightarrow (G(q,\theta), H(q,\theta)) \Longleftrightarrow \theta$$

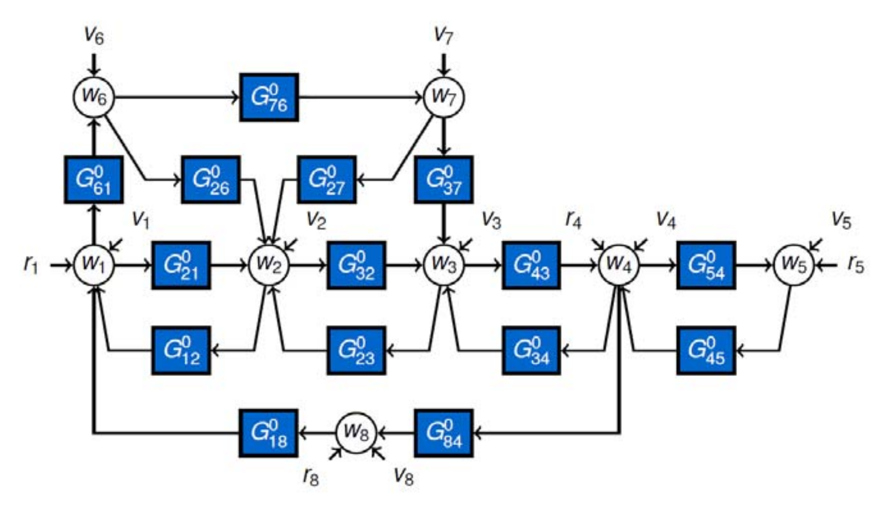
Classically: **trivial** **identifiability**

Network situation: **Nontrivial**

Reason:

- Freedom in network structure
- Freedom in presence of excitation and disturbances

Network Setup



Assumptions:

- Total of L nodes
- Network is well-posed and stable
- All $w_m, m = 1, \cdots L,$ and present r_m are measured
- Modules may be unstable
- Modules are strictly proper (can be generalized)

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & G^0_{12} & \cdots & G^0_{1L} \\ G^0_{21} & 0 & \ddots & G^0_{2L} \\ \vdots & \ddots & \ddots & \vdots \\ G^0_{L1} & G^0_{L2} & \cdots & 0 \end{bmatrix}}_{G^0} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R^0 \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H^0 \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix}$$

Network Setup

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & G^0_{12} & \cdots & G^0_{1L} \\ G^0_{21} & 0 & \ddots & G^0_{2L} \\ \vdots & \ddots & \ddots & \vdots \\ G^0_{L1} & G^0_{L2} & \cdots & 0 \end{bmatrix}}_{G^0} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R^0 \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H^0 \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix}$$

Different situations:

- ***p=L:*** Full rank noise process that disturbs every measured node
- ***p<L:*** Singular noise process, with the distinct options:
    a) All nodes are noise disturbed
    b) Some nodes noise-free; other nodes have full rank noise

Network Setup

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & G_{12}^0 & \cdots & G_{1L}^0 \\ G_{21}^0 & 0 & \ddots & G_{2L}^0 \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1}^0 & G_{L2}^0 & \cdots & 0 \end{bmatrix}}_{G^0} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R^0 \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H^0 \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix}$$

Different situations:

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Common situation in PE identification:
H^0 square and monic

Non-common situation:
H^0 non-square

Network identification setup

Network predictor:

$$\hat{w}(t|t-1) = \mathbb{E}\{w(t) \mid w^{t-1}, r^t\}$$

with $w^{t-1} = \{w(0), w(1), \cdots w(t-1)\}$

The network is defined by: (G^0, R^0, H^0)

a network model is denoted by: $M = (G, R, H)$

and a network model set by:

$$\mathcal{M} = \{M(\theta) = (G(\theta), R(\theta), H(\theta)), \theta \in \Theta\}$$

Models manifest themselves in identification through their predictor

Network identifiability

Decompose the node signals

$$w(t) = \begin{bmatrix} w_a(t) \\ w_b(t) \end{bmatrix}$$

with $w_a(t)$ noisy and $w_b(t)$ noise-free (a priori known).

Network identification setup

Problem with noise free-nodes:

$$\hat{w}(t|t-1) = W(q) \begin{bmatrix} w(t) \\ r(t) \end{bmatrix}$$

Filter $W(q)$ is non-unique, due to the noise-free nodes in $w(t)$

The predictor filter can be made unique when removing the noise-free signals as inputs

$$\begin{cases} \hat{w}(t|t-1) = P(q) \begin{bmatrix} w_a(t) \\ r(t) \end{bmatrix} \\ \hat{w}_b(t|t-1) = w_b(t) \end{cases}$$

Network identifiability

When can network models be distinguished through identification?

Two optional directions to continue:

1. The philosophical path (Plato)

2. The pragmatic path (Aristoteles)

Network identifiability (Philosphical path)

Generalized notion:

Consider an identification criterion $\mathcal{J}$ determining:

$$J(z, \mathcal{M})$$

with z measured data, $\mathcal{M}$ a model set,
and $J(z, \mathcal{M})$ the solution set of the identification

Then model set $\mathcal{M}$ is network identifiable (w.r.t. J) at M_0 if in $\mathcal{M}$ there does not exist a model $M_1 \neq M_0$ that always appears together with M_0 in $J(z, \mathcal{M})$

$\mathcal{M}$ is network identifiable (w.r.t. J) if it is network identifiable at all $M_0 \in \mathcal{M}$

Van den Hof, 1989, 1994 Weerts et al. (2016)

Network identifiability

Identification criterion for the situation of noise-free nodes:

The identification criterion:

$$J\left(\mathrm{z},\mathcal{M}\right)=\left\{\begin{array}{l}\underset{M(\theta)}{\arg}\ \underset{\theta\in\Theta}{\min}\ \bar{\mathbb{E}}\,\varepsilon_a^T(t,\theta)\Lambda^{-1}\varepsilon_a(t,\theta)\\ \text{subject to: } \varepsilon_b(t,\theta)=0 \quad \text{for all } t.\end{array}\right\}$$

Noise-free nodes can be predicted exactly

Network identifiability (merging of the paths)

Theorem 1 *(or Definition 1)*

Denote $T(q)$ as the transfer function $\begin{pmatrix} e \\ r \end{pmatrix} \rightarrow w$

$$T(q) = (I - G(q)^{-1}U(q) \quad \text{with} \quad U(q) := \begin{bmatrix} H_a(q) & R_a(q) \\ 0 & R_b(q) \end{bmatrix}$$

and let $T(q,\theta)$ be its parametrized version

Then the network model set $\mathcal{M}$ is network identifiable at $M_0 = M(\theta_0)$ (w.r.t. J) if for all models $M(\theta_1) \in \mathcal{M}$:

$$T(q,\theta_1) = T(q,\theta_0) \Longrightarrow M(\theta_1) = M(\theta_0)$$

Goncalves and Warnick, 2008; Weerts et al, SYSID2015; Weerts et al. ALCOSP 2016; Gevers and Bazanella, 2016.

Network identifiability

$\mathcal{M}$ is network identifiable (w.r.t. J) if it is network identifiable at all models $M \in \mathcal{M}$.

Question of identifiability means:

Can the models in a network model set be distinguished from each other through identification

a) With respect to a single model $M_0 = M(\theta_0)$
b) With respect to all models in the set

Network identifiability

Theorem 2

$\mathcal{M}$ is network identifiable at $M_0 = M(\theta_0)$ (w.r.t. J) if there exists a square, nonsingular and fixed $Q(q)$ such that

$$U(q,\theta)Q(q) = \begin{bmatrix} D(q,\theta) & F(q,\theta) \end{bmatrix}$$

With $D(q,\theta)$ square, and such that $D(q,\theta)$ is diagonal and full rank for all

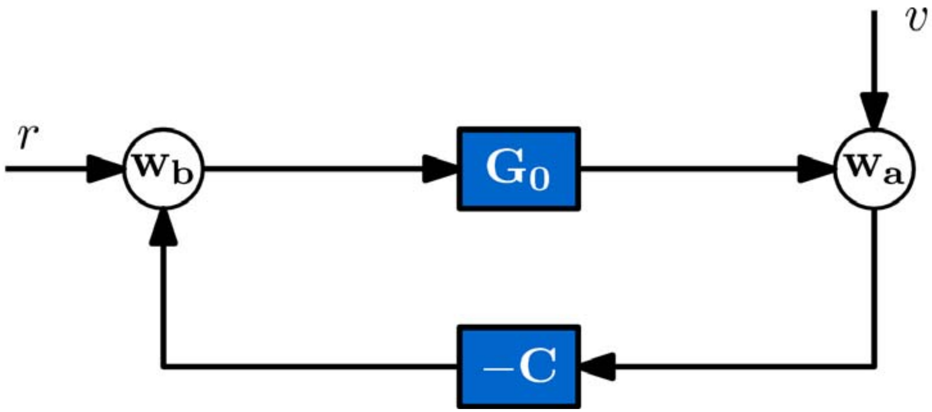
$$\theta \in \Theta_0 := \{\theta \in \Theta \mid T(q,\theta) = T(q,\theta_0)\}$$

$\mathcal{M}$ is network identifiable (w.r.t. J) if the above holds for $\Theta_0 = \Theta$

Goncalves and Warnick, 2008; Weerts et al, SYSID2015; Gevers and Bazanella, 2016; Weerts et al., ArXiv 2016;

Closed-loop example

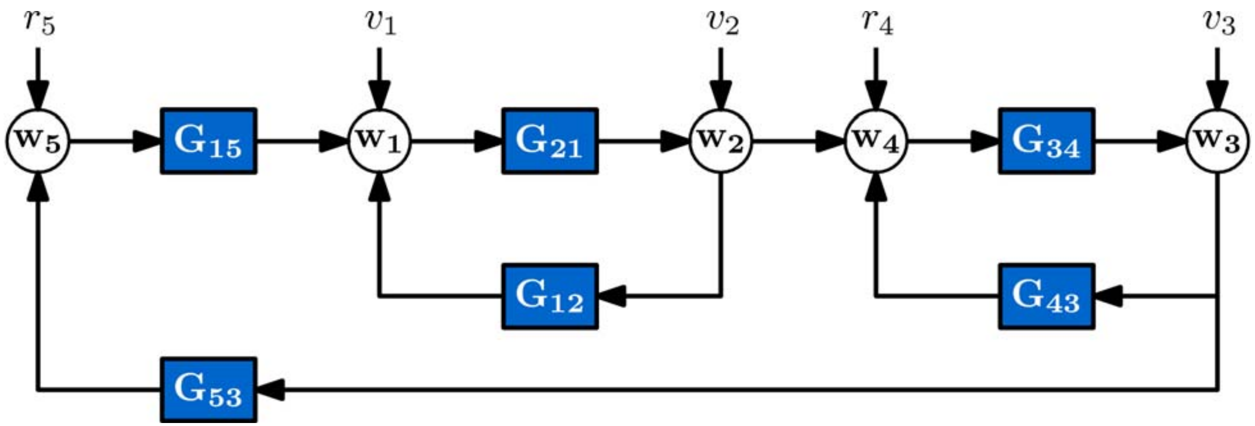
This classical closed-loop system has a noise-free node



$$\mathcal{M} \text{ with } G(\theta) = \begin{bmatrix} 0 & G_0(\theta) \\ -C(\theta) & 0 \end{bmatrix}, \; H(\theta) = \begin{bmatrix} H_a(\theta) \\ 0 \end{bmatrix}, \; R = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} H(\theta) & R(\theta) \end{bmatrix}$ **is diagonal and full rank → identifiability**

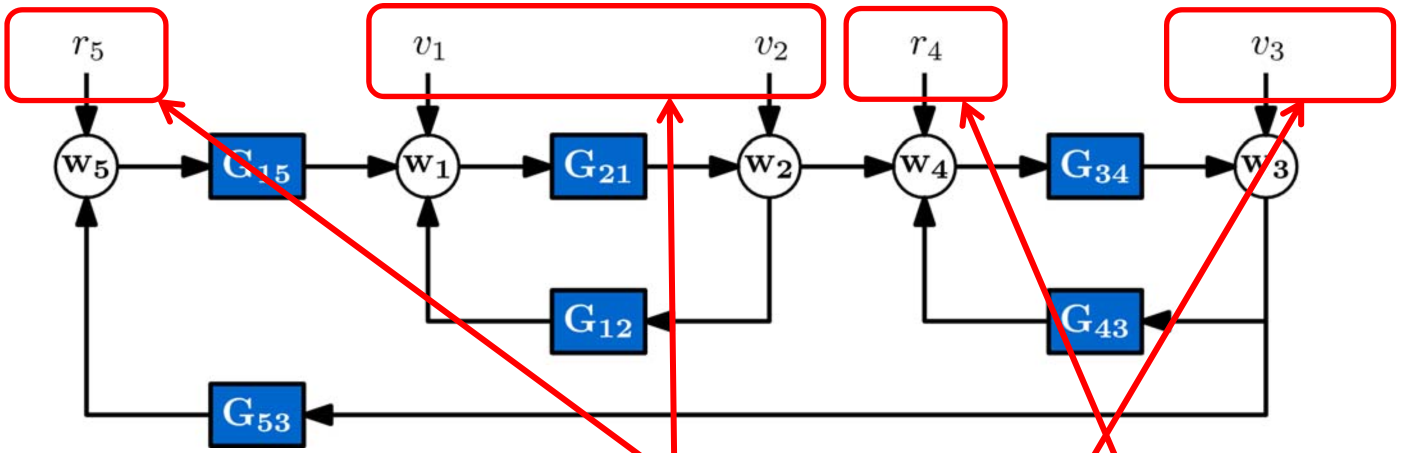
Example correlated noises



There are noise-free nodes, and $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are expected to be correlated

$$\mathcal{M} \text{ with } H(\theta) = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) & 0 \\ H_{21}(\theta) & H_{22}(\theta) & 0 \\ 0 & 0 & H_{33}(\theta) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example correlated noises



There are noise-free nodes, and v_1 and v_2 are expected to be correlated

$$\mathcal{M} \text{ with } H(\theta) = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) & 0 \\ H_{21}(\theta) & H_{22}(\theta) & 0 \\ 0 & 0 & H_{33}(\theta) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can not arrive at a diagonal structure in $\begin{bmatrix} H(\theta) & R(\theta) \end{bmatrix}$

Network identifiability

Observations:

a) A simple test can be performed to check the condition

b) The condition is typically fulfilled if each node w_j is excited by either an external excitation r_j or a noise v_j that are uncorrelated with the external signals on other nodes.

c) The result is rather **conservative**:
   1. Restricted to situation where $U(q,\theta)$ is full row rank
   2. Does not take account of structural properties of $G(q,\theta)$ e.g. modules/controllers that are known a priori

Network identifiability

Theorem 3 – identifiability in case of structure restrictions

Assumptions:

a) Each parametrized entry in $M(q,\theta)$ covers the set of all proper rational transfer functions
b) All parametrized elements in $M(q,\theta)$ are parametrized independently

Then the network model set $\mathcal{M}$ is network identifiable at $M_0 = M(\theta_0)$ (w.r.t. J) if and only if:

- Each row i of $\begin{bmatrix} G(\theta) & U(\theta) \end{bmatrix}$ has at most *K*+*p* parametrized entries
- For each row i, $\breve{T}_i(q,\theta_0)$ has full row rank

where: $\breve{T}_i(q,\theta_0)$ is the submatrix of $T(q,\theta_0)$, composed of those rows *j* that correspond to elements $G_{ij}(q,\theta)$ that are parametrized

Goncalves and Warnick, 2008; Weerts et al., ArXiv 2016;

Network identifiability

Theorem 3 – identifiability in case of structure restrictions

Assumptions:

a) Each parametrized entry in $\boldsymbol{M(q,\theta)}$ covers the set of all proper rational transfer functions

b) All parametrized elements in $\boldsymbol{M(q,\theta)}$ are parametrized independently

Then the network model set $\mathcal{M}$ is network identifiable (w.r.t. $\boldsymbol{J}$) if and only if:

- Each row i of $\begin{bmatrix}\boldsymbol{G(\theta)} & \boldsymbol{U(\theta)}\end{bmatrix}$ has at most *K+p* parametrized entries
- For each row i, $\boldsymbol{\check{T}_i(q,\theta)}$ has full row rank for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$

where: $\boldsymbol{\check{T}_i(q,\theta_0)}$ is the submatrix of $\boldsymbol{T(q,\theta_0)}$, composed of those rows *j* that correspond to elements $\boldsymbol{G_{ij}(q,\theta)}$ that are parametrized

Network identifiability

Corollary – situation of $U(\theta)$ full row rank

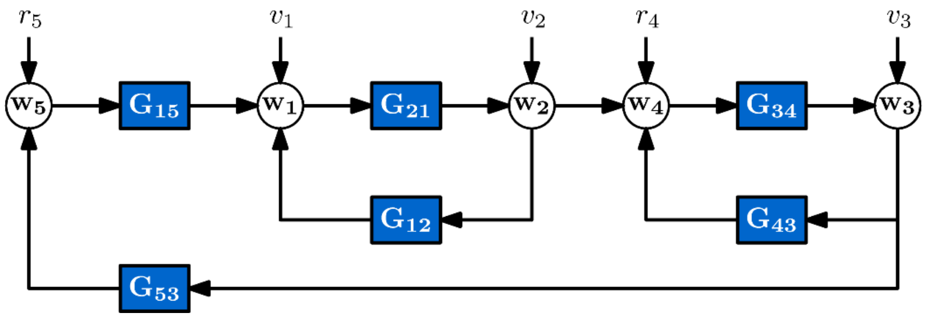
If $U(\theta)$ is full row rank for $(\theta = \theta_0 \;/\; \forall\theta \in \Theta)$

Then $\mathcal{M}$ is network identifiable (at $M(\theta_0)$) if and only if:

- Each row i of $\begin{bmatrix} G(\theta) & U(\theta) \end{bmatrix}$ has at most *K+p* parametrized entries

The number of parametrized transfer functions that map into a node w_i should not exceed the total number of excitation+noise signals in the network.

Example correlated noises (continued)



If we restrict the structure of $G(\theta)$:

$$G(\theta) = \begin{bmatrix} 0 & G_{12}(\theta) & 0 & 0 & G_{15}(\theta) \\ G_{21}(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{34}(\theta) & 0 \\ 0 & 1 & G_{43}(\theta) & 0 & 0 \\ 0 & 0 & G_{53}(\theta) & 0 & 0 \end{bmatrix} \quad U(\theta) = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) & 0 & 0 & 0 \\ H_{21}(\theta) & H_{22}(\theta) & 0 & 0 & 0 \\ 0 & 0 & H_3(\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

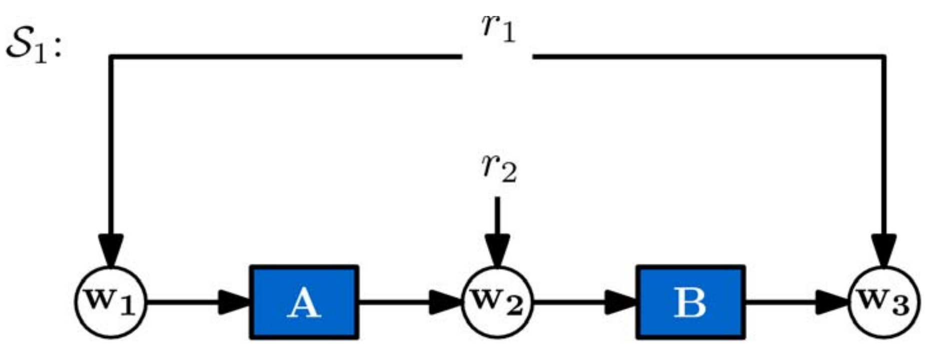
Node/row 1 has 4 unknowns < K+p = 5

Node/row 2 has 3 unknowns (2 from noise model) < K+p

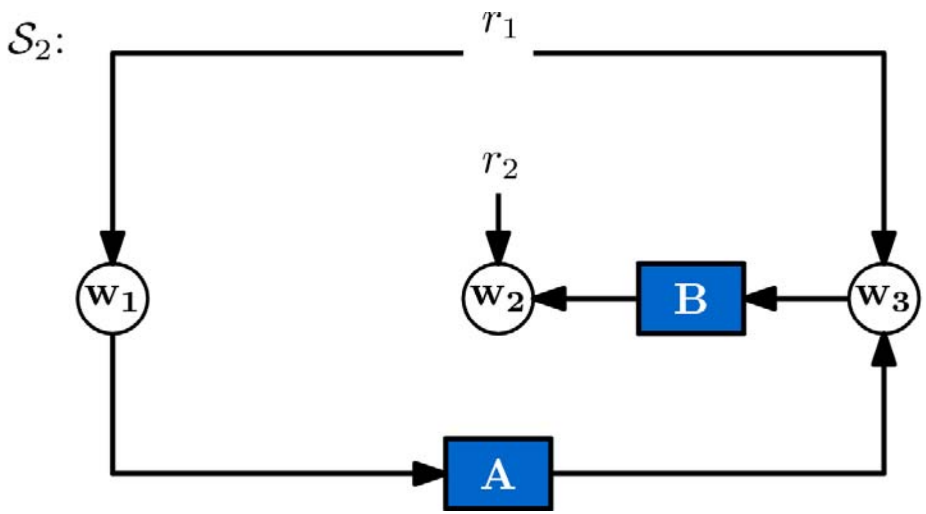
→ **identifiable!**

Example: identifiability at a particular model

System 1



System 2



$$\boldsymbol{T : r \rightarrow w}$$

$$T_1 = \begin{bmatrix} 1 & 0 \\ A & 1 \\ AB+1 & B \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 \\ (A+1)B & 1 \\ A+1 & 0 \end{bmatrix}$$

Example: identifiability at a particular model

$$\mathcal{M} \text{ with } G(\theta) = \begin{bmatrix} 0 & G_{12}(\theta) & G_{13}(\theta) \\ G_{21}(\theta) & 0 & G_{23}(\theta) \\ G_{31}(\theta) & G_{32}(\theta) & 0 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Can we find a unique model that satisfies $T = (I - G(\theta))^{-1} R$

$$(I - G(\theta))T_1 = R \Rightarrow \begin{cases} G_{12} = G_{13} = G_{23} = G_{31} = 0 \\ G_{21} = A \\ G_{32} = B \end{cases}$$

Unique

$$(I - G(\theta))T_2 = R \Rightarrow \begin{cases} G_{12} = G_{13} = G_{32} = 0 \\ G_{31} = A \\ G_{21} = (A+1)(B - G_{23}) \end{cases}$$

Non-unique

Uniqueness of the solution depends on the system

The model set is network identifiable in system 1 but not in system 2

Result

When is the model identifiable? Evaluate: $(I - G(\theta))T = R$

Condition 1

At most *K*+*p* parameterized transfer functions

K+*p*=2 equations

$$\begin{bmatrix} 1 & -G_{12}(\theta) & -G_{13}(\theta) \\ -G_{21}(\theta) & 1 & -G_{23}(\theta) \\ -G_{31}(\theta) & -G_{32}(\theta) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A+1)B & 1 \\ A+1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Condition 2

$$\begin{bmatrix} 1 & -G_{12}(\theta) & -G_{13}(\theta) \\ -G_{21}(\theta) & 1 & -G_{23}(\theta) \\ -G_{31}(\theta) & -G_{32}(\theta) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A+1)B & 1 \\ A+1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Appropriate sub-matrix $\check{T}_1$ full row rank

Example 1 continued

The reason there is no identifiability

$$\begin{bmatrix} 1 & -G_{12}(\theta) & -G_{13}(\theta) \\ -G_{21}(\theta) & 1 & -G_{23}(\theta) \\ -G_{31}(\theta) & -G_{32}(\theta) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A+1)B & 1 \\ A+1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Appropriate sub-matrix $\breve{T}_2$ **not** full row rank → not identifiable

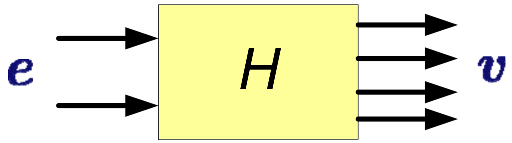
Summary

- Concept of network identifiability has been introduced and extended beyond the classical PE assumptions (all measurements noisy) "can models be distinguished in identification?"

- The network transfer functions T remain the objects that can be uniquely identified from data

- Results lead to verifiable conditions on the network structure / parametrization / presence of external signals

- The framework is fit for extending it to the general situation of singular / reduced-rank noise



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